"DIFFERENTIAL EQUATION"

DATE 27/03/23

UNIT - 3

★ **Differential equation** - Jismai derivative $\left(\frac{dy}{dx}\right)$ ho..

► Ex. -

① $dy/dx + y = 0$

② $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^x$

Firstly Say "order" then "degree"

★ **ORDER of a D.E.** - Highest $\left(\frac{dy}{dx}\right)$.

(i) $\frac{d^2y}{dx^2} + y = 0 \rightarrow$ (order - 2)

(ii) $\frac{d^3y}{dx^3} + 4y = 0 \rightarrow$ (order - 3)

★ **Degree of a D.E.** - Hightest $\left(\frac{dy}{dx}\right)$ power Ki Value.

(i) $\left(\frac{d^2y}{dx^2}\right)^1 + \left(\frac{dy}{dx}\right)^3 + y = 0 \rightarrow$ (deg - 1)

(ii) $\left(\frac{d^3y}{dx^3}\right)^1 + \frac{dy}{dx} + y = 0 \rightarrow$ (deg - 1)

★ Differential eqn of 1st order & 1st degree :-

▸ [METHOD - 1] ⇒ Separation of Variable :-

Que. 1 $\frac{dy}{dx} = \frac{y}{x}$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\log y = \log(x) + \log C$

$\cancel{\log}(y) = \cancel{\log}(xC)$

$\boxed{y = xC}$ Ans.

Que. 2 $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

[∘ $\tan^{-1} x = \int \frac{dx}{1+x^2}$]

$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + C$

$\tan^{-1} y = \tan^{-1} x + \tan^{-1} C$

$\cancel{\tan^{-1}} y = \cancel{\tan^{-1}} \left(\frac{x + C}{1 - xC} \right)$

$\boxed{y = \frac{x + C}{1 - xC}}$ Ans.

[∘ Formula Used

∘∘ $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$]

[METHOD - 2] ⇒ Linear Differential EQUATION -

Jinki $\left(\frac{dy}{dx}\right)^{1}$ ki power/degree one ho.

y - form linear	x - form linear
$\boxed{\frac{dy}{dx} + Py = Q}$	$\boxed{\frac{dx}{dy} + Px = Q}$
x (OR) May be Constant	y (OR) May be Constant
$\boxed{I.F = e^{\int P dx}}$	$\boxed{I.F. = e^{\int P dy}}$
$\boxed{y(I.F.) = \int Q(I.F.) dx + C}$	$\boxed{x(I.F.) = \int Q(I.F.) dy + C}$

Que.1 $\frac{dy}{dx} + \frac{y}{x} = x \Rightarrow \boxed{P = \frac{1}{x}}, \boxed{Q = x}$

$$I.F. = e^{\int P dx}$$

$$I.F. = e^{\int \frac{1}{x} dx}$$

$$I.F. = e^{\log x}$$

$$I.F. = x^{\log_e e}$$

$$\boxed{I.F. = x}$$

$$y(I.F.) = \int Q.(I.F.) dx + C$$

$$y.(x) = \int x(x) dx + C$$

$$yx = \int x^2 dx + C$$

$$yx = \frac{x^3}{3} + C$$

$$\boxed{y = \frac{x^2}{3} + \frac{C}{x}}$$ Ans.

Que.2 $\frac{dy}{dx} + \frac{y}{x} = e^x$

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$\boxed{P = \frac{1}{x}}$ $\boxed{Q = e^x}$

$I.F. = e^{\int P dx}$

$I.F. = e^{\int \frac{1}{x} dx}$

$I.F. = e^{\log x}$

$I.F. = x^{\log_e e^1}$

$\boxed{I.F. = x}$

$y(I.F) = \int Q(I.F) dx + C$

$y(x) = \int e^x (x) dx + C$

$xy = \int x e^x dx + C$

▸ Integration By Parts

$$\boxed{\int I \cdot II \, dx = I \int II \, dx - \int \left[\frac{dI}{dx} \left(\int II \, dx \right) \right] dx}$$

Lograth. → Trigono.

ILATE → Select I function by this Rule!

Inverse Algebra Exponential

$xy = \int \underset{I}{x} \underset{II}{e^x} dx + C$

$xy = x \int e^x dx - \int \left[\frac{dx}{dx} \int e^x dx \right] dx + C$

$xy = x e^x - \int e^x dx + C$

$xy = x e^x - e^x + C$

$\boxed{xy = e^x (x-1) + C}$ Ans.

Que.3 $\frac{dy}{dx} + \frac{y}{x} = e^{x^2} \Rightarrow \boxed{P = \frac{1}{x}}$ $\boxed{Q = e^{x^2}}$

$$\boxed{y(I.F.) = \int Q(I.F.)\,dx + C}$$

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$$I.F. = e^{\int P\,dx}$$

$$I.F. = e^{\int \frac{1}{x} dx}$$

$$I.F. = e^{\log x}$$

$$\boxed{I.F. = x}$$

$$y(x) = \int e^{x^2}(x)\,dx + C$$

$$xy = \int x \cdot e^{x^2}\,dx + C$$

Let, $\boxed{x^2 = t}$

Diffren. both side w.r.t (x)

$$\frac{d(x^2)}{dx} = \frac{dt}{dx}$$

$$\boxed{2x = \frac{dt}{dx}} \qquad \boxed{x\,dx = \frac{dt}{2}}$$

$$xy = \int e^t \frac{dt}{2} + C$$

$$xy = \frac{1}{2}\int e^t\,dt + C$$

$$xy = \frac{1}{2}\left(\frac{e^{t^2}}{2}\right) + C$$

$$\boxed{xy = \frac{e^{x^4}}{4} + C}$$

Ans.

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Que.1 $(1+y^2)dx = (\tan^{-1}y - x)dy$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x} \Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{x}{1+y^2} + \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2}$$

$$\boxed{\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}} \qquad \boxed{\frac{dx}{dy} + Px = Q}$$

$$\boxed{P = \frac{1}{1+y^2}} \qquad \boxed{Q = \frac{\tan^{-1}y}{1+y^2}}$$

$$I.F = e^{\int P\,dy}$$

$$I.F. = e^{\int \frac{1}{1+y^2}dy}$$

$$\boxed{I.F. = e^{\tan^{-1}y}}$$

$$\Rightarrow \boxed{x(I.F.) = \int Q(I.F.)\,dy + C}$$

$$x(e^{\tan^{-1}y}) = \int \frac{\tan^{-1}y}{1+y^2}.(e^{\tan^{-1}y})dy + C$$

Let, $\boxed{\tan^{-1}y = t}$ → diffren. w.r.t (y) $\Rightarrow \frac{d(\tan^{-1}y)}{dy} = \frac{dt}{dy}$

$$\frac{1}{1+y^2} = \frac{dt}{dy}$$

$$\boxed{\frac{dy}{1+y^2} = dt}$$

$$xe^t = \int t.e^t \cancel{dy}\,dt + C$$

$$x.e^t = \left[e^t(t-1)\right] + C$$

Replace $\boxed{t = \tan^{-1}y}$ Divide by this

Ans. $\boxed{x = \tan^{-1}y - 1 + \frac{C}{e^{\tan^{-1}y}}}$

H.W Que.

Que. 2 $\cos^2 x \frac{dy}{dx} + y = \tan x$ $\boxed{P = \sec^2 x}$ $\boxed{Q = \frac{\tan x}{\cos^2 x}}$

Divide whole eq.ⁿ by $\cos^2 x$

$$\boxed{\frac{dy}{dx} + y \cdot \sec^2 x = \frac{\tan x}{\cos^2 x}}$$

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$\Rightarrow y(I.F.) = \int Q(I.F.)\,dx + C$

$I.F. = e^{\int P dx}$

$I.F. = e^{\int \sec^2 x dx}$

$\boxed{I.F. = e^{\tan x}}$

$y(e^{\tan x}) = \int \frac{\tan x}{\cos^2 x} \cdot (e^{\tan x})\,dx + C$

$\Rightarrow$ Let $\boxed{t = \tan x}$

diffren. both side by $dx \longrightarrow \frac{dt}{dx} = \frac{d(\tan x)}{dx}$

$\frac{dt}{dx} = \sec^2 x$

$\boxed{dt = \sec^2 x\, dx}$

$y \cdot e^t = \int \cancel{\tan x}\, t \cdot e^t\, dt + C$

$y \cdot e^t = e^t(t-1) + C$

Replace (t) with $\tan x$

$$\boxed{y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + C}$$ Ans.

★ Bernqullis equation :-

$$\boxed{\frac{dy}{dx} + py = Q \cdot y^n}$$ Divide by y^n.

$$\boxed{\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q} \quad \text{— (1)}$$

$$\frac{1}{y^{n-1}} = v \Rightarrow \boxed{y^{1-n} = V}$$

Diffren. w.r.t (x)

$$\frac{d(y^{1-n})}{dx} = \frac{dv}{} \Rightarrow (1-n)\, y^{1-n-1}\frac{dy}{dx} = \frac{dv}{dx}$$

Chain Rule Apply

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$$

Put $\frac{1}{y^{n-1}}$ & $\frac{1}{y^n}\frac{dy}{dx}$ put in eq. (1)

Solve it ← Linear

Que. 1 $xy(1+xy^2)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{xy + x^2y^3} \Rightarrow \frac{dx}{dy} = xy + x^2y^3$

$\frac{dx}{dy} - xy = x^2y^3 \Rightarrow \boxed{P = -y}$ $\boxed{Q = x^2y^3}$

Divided by x^2

$\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$

Put $\boxed{v = \frac{1}{x}}$ & $\boxed{\frac{1}{x^2}\frac{dx}{dy} = -\frac{dv}{dy}}$

$-\frac{dv}{dy} - v.y = y^3 \Rightarrow \boxed{\frac{dv}{dy} + v.y = -y^3}$

$P = y$ $Q = -y^3$

$I.F. = \int e^{P dy} = \int e^{\int y dy} = e^{y^2/2}$

$\boxed{I.F. = e^{y^2/2}}$

$v(I.F.) = \int Q(I.F.)\, dy + C$

$v.(e^{y^2/2}) = \int -y^3.(e^{y^2/2})\, dy + C$

Let $\boxed{\frac{y^2}{2} = t}$

diff. w.r.t dy

$\frac{1}{2}\frac{d(y^2)}{dy} = \frac{dt}{dy} \Rightarrow \boxed{y = \frac{dt}{dy}}$

$v.e^t = -\frac{2}{2}\int y^2.y\, e^t dy + C$

$v.e^t = -2\int t\,dt + C$

$v.e^t = -2e^t(t-1) + C$

Replace $\boxed{v = \frac{1}{x}}$ & $\boxed{t = \frac{y^2}{2}}$

$\frac{1}{x}.e^{y^2/2} = -2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + C$

Ans.

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H.W Que.

Que.2 $\frac{dy}{dx} + x\sin^2 y = x^3\cos^2 y$

Divided by $\cos^2 y$

$\sec^2 y \frac{dy}{dx} + x\tan^2 y = x^3$

Divided by $\sec^2 x$

★ Exact differential Eq.^{n} :-

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\cancel{\partial y}\,\partial x} \quad \text{[Partial differentiation]}$$

Sol.^{n} => $$\boxed{\int_{y} M\,dx + \int_{x} N\,dy = C}$$

Que. 1

$$(x^2 - ay)\,dx = (ax - y^2)\,dy$$

$$(x^2 - ay)\,dx - (ax - y^2)\,dy = 0 \quad \text{— (1)}$$

$$\Rightarrow M(x,y) = x^2 - ay$$

$$\Rightarrow N(x,y) = -(ax - y^2) = y^2 - ax$$

$$\boxed{\frac{\partial M}{\partial y} = -a}\ , \quad \boxed{\frac{\partial N}{\partial x} = -a}$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$ Given diff eq. (1) is =>

$$\int_{y\ \text{const.}} M\,dx + \int_{x\ \text{Const.}} N\,dy = C$$

$$\int_{y\ \text{Const.}} (x^2 - ay)\,dx + \int_{x\ \text{Const.}} (y^2 - ax)\,dy = C$$

$$\frac{x^3}{3} - \cancel{ayx} + \frac{y^3}{3} - \cancel{axy} = C$$

$$\boxed{\frac{x^3}{3} + \frac{y^3}{3} = C}$$ Ans.

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★ Rule - I -

$\boxed{Mdx + Ndy = 0}$ be a homogeneous equation then If is $\frac{1}{Mx + Ny}$. $\boxed{Mx + Ny \neq 0}$

Ex.- $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — ①

$M = (x^2y - 2xy^2)$
$N = -(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy$, $\frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$$I.F. = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)x + [-(x^3 - 3x^2y)]y}$$

$$\boxed{I.F. = \frac{1}{x^2y^2}}$$

From eq. ① × I.F.

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{— ②}$$

$M = \left(\frac{1}{y} - \frac{2}{x}\right)$, $N = \left(\frac{x}{y^2} - \frac{3}{y}\right)$

$$\frac{\partial M}{\partial y} = \frac{1}{y^2} \quad , \quad \frac{\partial N}{\partial x} = \frac{-1}{y^2}$$ ★

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M dx + \int N dy = C$$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) \partial x + 3\int \frac{1}{y} \partial y = C$$

$$\boxed{\frac{x}{y} - 2\log x + 3\log y = C}$$ Ans.

▶ RULE - II :-

$Mdx + Ndy = 0$ is of line form

$$\boxed{F_1(x,y)y\,dx + F_2(x,y)x\,dy = 0}$$ then

$$\boxed{I.F. = \frac{1}{Mx \cdot Ny}}$$, $Mx - Ny \neq 0$

▶ RULE - III :-

$$\boxed{Mdx + Ndy = 0}$$ then

$\frac{1}{N}\left(\frac{dM}{dy} - \frac{\partial N}{\partial x}\right)$ is a function of x.

then $$\boxed{I.F. = e^{\int f(x)dx}}$$

- **RULE - IV** -

$$\boxed{M\,dx + N\,dy = 0}$$ then

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$ is a function of y

then $$\boxed{I.F. = e^{\int f(y)\,dy}}$$

- Ex. →

$$\left(xy^2 - e^{1/x^3}\right)dx - x^2y\,dy = 0 \quad -(1)$$

$$M = \left(xy^2 - e^{1/x^3}\right)$$
$$N = -x^2y$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

$$I.F. = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$I.F. = \frac{1}{-x^2y}\left(2xy + 2xy\right) = \boxed{-\frac{4}{x} = f(x)}$$

$$\Rightarrow I.F. = e^{\int f(x)\,dx} = e^{-\int \frac{4}{x}dx} = e^{-4\log x}$$

$$\boxed{I.F. = \frac{1}{x^4}}$$ Ans.

H.W Que

Ex. - $$\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx - \frac{y}{x^2}\,dy = 0$$